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ELECTROMAGNETIC SPINOR AND WAVE FUNCTIONS IN MINKOWSKI SPACETIME

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Abstract
We show how generate the electromagnetic spinor verifying the Maxwell equations in vacuum; we also exhibit the Whittaker and Bateman wave functions and their connection with electromagnetic fields in Minkowski geometry.

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All electromagnetic information is contained in the Faraday's skew-symmetric tensor [1-3]:

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{pmatrix}, \tag{1}$$

where $\vec{E} = (E_1, E_2, E_3)$ and $\vec{B} = (B_1, B_2, B_3)$ are the electric and magnetic fields expressed in the MKS system of units, respectively, verifying the Maxwell equations in empty spacetime:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}. \tag{2}$$

If we introduce the Riemann [4]-Silberstein [5, 6] complex vector (thus named by Bialynicki-Birula [7]) [8]:

$$\vec{F} = c\,\vec{B} + i\,\vec{E}, \tag{3}$$

then (2) are equivalent to two complex equations [9]:

$$\vec{\nabla} \cdot \vec{F} = 0, \qquad \vec{\nabla} \times \vec{F} + \frac{i}{c}\frac{\partial}{\partial t}\vec{F} = \vec{0}, \tag{4}$$

and the corresponding spinorial form is given by [10-14]:

$$\partial_{B\dot{C}}\, \varphi^{AB} = 0, \tag{5}$$

involving the symmetric Maxwell spinor. It is clear that, without boundary conditions, the equations (5) have many solutions, thus it is important to have a method to construct them.

On the other hand, we know that a unitary quaternion [15-17] generates rotations in three and four dimensions [18-20], but, what happens with a null quaternion? Here we consider the following case:

$$\mathbf{A} = -\sin\tau - i\,\mathbf{I} + \cos\tau\,\mathbf{K} \quad \therefore \quad \mathbf{A}\overline{\mathbf{A}} = (\sin\tau)^2 + i^2 + (\cos\tau)^2 = 0, \quad (6)$$

with its associated 2x2 complex matrix:

$$\widetilde{M} = \begin{pmatrix} ie^{i\tau} & 1 \\ 1 & -ie^{-i\tau} \end{pmatrix}. \quad (7)$$

Now the interesting result is that (7) allows construct solutions for (5), in fact:

$$(\varphi^{AB}) = \frac{1}{2}\int_0^{2\pi} \widetilde{M}\ G(u, v, \tau)\, d\tau, \qquad u = x\cos\tau + y\sin\tau + i\,z,$$
$$v = x\sin\tau - y\cos\tau + c\,t, \quad (8)$$

where G is an arbitrary function of its arguments; hence (8) is a factory to elaborate solutions for the Maxwell equations in vacuum, that is:

$$\varphi^{11} = \frac{i}{2}\int_0^{2\pi} e^{i\tau}\, G\, d\tau, \qquad \varphi^{12} = \varphi^{21} = \frac{1}{2}\int_0^{2\pi} G\, d\tau,$$
$$\varphi^{22} = -\frac{i}{2}\int_0^{2\pi} e^{-i\tau}\, G\, d\tau\,. \quad (9)$$

We have the expressions for the spinor covariant derivative [12]:

$$\partial_{1\dot{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{c\,\partial t} + \frac{\partial}{\partial z}\right), \quad \partial_{2\dot{1}} = \overline{\partial_{1\dot{2}}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$
$$\partial_{2\dot{2}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{c\,\partial t} - \frac{\partial}{\partial z}\right), \quad (10)$$

then it is simple to prove the relations:

$$\partial_{1\dot{1}}G = \frac{1}{\sqrt{2}}\left(\frac{\partial G}{\partial v} + i\,\frac{\partial G}{\partial u}\right) = i\,e^{-i\tau}\,\partial_{2\dot{1}}G\,,\ \partial_{2\dot{2}}G = \frac{1}{\sqrt{2}}\left(\frac{\partial G}{\partial v} - i\,\frac{\partial G}{\partial u}\right) =$$
$$-i\,e^{i\tau}\,\partial_{1\dot{2}}G\,, \quad (11)$$

thus with (9) and (11) is immediate verify (5), *q.e.d.*

The components of the Riemann-Silberstein complex vector (3) are given by:

$$F_1 = i(\varphi^{22} - \varphi^{11}) = \int_0^{2\pi} G \cos\tau \, d\tau, \qquad F_2 = -(\varphi^{22} + \varphi^{11}) = \int_0^{2\pi} G \sin\tau \, d\tau, \; F_3 = 2i\varphi^{12} = i\int_0^{2\pi} G \, d\tau,$$

in agreement with the result of Bateman [9]:

$$\vec{\mathrm{F}} = \int_0^{2\pi} G(x\cos\tau + y\sin\tau + i\,z, x\sin\tau - y\cos\tau + c\,t, \tau)\; \vec{\mathrm{R}}\, d\tau, \quad \vec{\mathrm{R}} = (\cos\tau, \sin\tau, i). \tag{12}$$

We have the valuable theorem of I. Robinson and J. L. Synge [2, 22-25]:

"Every solution of the vacuum Maxwell equations in Minkowski spacetime can be written in terms of two real wave functions and a constant real bivector", (13)

and the corresponding Faraday tensor is given by the expression:

$$\mathrm{F}_{\mu\nu} = (\mathrm{H}_\mu{}^\lambda\, \mathrm{U}_{,\lambda} + {}^*\mathrm{H}_\mu{}^\lambda\, \mathrm{V}_{,\lambda})_{,\nu} - (\mathrm{H}_\nu{}^\lambda\, \mathrm{U}_{,\lambda} + {}^*\mathrm{H}_\nu{}^\lambda\, \mathrm{V}_{,\lambda})_{,\mu}\,, \tag{14}$$

where $\Box\mathrm{U} = \Box\mathrm{V} = 0$ and ${}^*\mathrm{H}_{\mu\nu}$ is the dual of the constant skew-symmetric tensor $\mathrm{H}_{\mu\nu}$.

The first person to suggest that a vacuum electromagnetic field could be constructed from only two real wave functions was Whittaker [23, 24, 26-29]: He showed explicitly that the Liénard-Wiechert field of an accelerating point charge could be derived from a pair of wave functions, and he calculated these functions. With the constant tensor:

$$(\mathrm{H}^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \therefore \quad ({}^*\mathrm{H}^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

the relations (1) and (14) imply the following expressions of Whittaker [26]:

$$\mathrm{E}_1 = \frac{\partial^2 \mathrm{U}}{\partial x\, \partial z} + \frac{1}{c}\frac{\partial^2 \mathrm{V}}{\partial y\, \partial t}, \qquad \mathrm{E}_2 = \frac{\partial^2 \mathrm{U}}{\partial y\, \partial z} - \frac{1}{c}\frac{\partial^2 \mathrm{V}}{\partial x\, \partial t}, \qquad \mathrm{E}_3 = \frac{\partial^2 \mathrm{U}}{\partial x^2} + \frac{\partial^2 \mathrm{U}}{\partial y^2}, \tag{16}$$

$$c\,\mathrm{B}_1 = -\frac{\partial^2 \mathrm{V}}{\partial x\, \partial z} + \frac{1}{c}\frac{\partial^2 \mathrm{U}}{\partial y\, \partial t}, \quad c\,\mathrm{B}_2 = -\frac{\partial^2 \mathrm{V}}{\partial y\, \partial z} - \frac{1}{c}\frac{\partial^2 \mathrm{U}}{\partial x\, \partial t}, \quad c\,\mathrm{B}_3 = \frac{\partial^2 \mathrm{V}}{\partial x^2} + \frac{\partial^2 \mathrm{V}}{\partial y^2}.$$

Now we need a systematic method to construct the scalar wave functions U and V, in fact [26]:

$$\mathrm{U} = \int_0^{\pi}\int_0^{2\pi} f(x \sin u\, \cos v + y \sin u \sin v + z \cos u + c\, t, u, v)\, du\, dv,$$
$$\tag{17}$$
$$\mathrm{V} = \int_0^{\pi}\int_0^{2\pi} g(x \sin u\, \cos v + y \sin u \sin v + z \cos u + c\, t, u, v)\, du\, dv,$$

where f and g are arbitrary functions of their arguments. Bateman [9, 30] showed an alternative approach to find wave functions in four dimensions:

$$\mathrm{W} = \int_0^{2\pi} G(x \cos \tau + y \sin \tau + i\, z, x \sin \tau - y \cos \tau + c\, t, \tau)\, d\tau, \tag{18}$$

being G an arbitrary function. Finally, we comment that Whittaker [9, 31] proved that the function:

$$w = \int_0^{2\pi} h(z + i\, x \cos \alpha + i\, y \sin \alpha, \alpha)\, d\alpha, \tag{19}$$

for h an arbitrary function, satisfies the Laplace equation $\nabla^2 w = 0$.

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